Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

A SYNTHETICAL DEMONSTRATION *of the* RULE *for the* QUADRATURE *of* SIMPLE CURVES, *in the Analysis per Equationes terminorum numero infinitas.*

By the Rev. M. YOUNG, *D. D.*

DOCTOR Wallis, about the year 1651, having met with the mathematical writings of Torricelli, in which, amongst other things, he explains Cavalerius's attempt to render the ancient method of Exhaustions more concise by his *Geometry* of indivisibles, conceived that an *Arithmetic* of infinites might be applied to the contemplation of curve lines with success; and that perhaps the quadrature of the circle, were it at all possible, might finally be attained in this manner. What led him to these expectations, as he tells us in his dedication to Oughtred, was this: the ratio of the infinite circles of a cone to as many of a cylinder on the same base and of the same altitude, Read March 6, 1786.

altitude, was known, viz. as 1 to 3; but all their diameters in the triangle through the axis of the cone, to as many in the parallelogram through the axis of the cylinder, as 1 to 2. In like manner it was known, that all the circles in the parabolic conoid were to as many circles in a cylinder as 1 to 2; but all the diameters of the former to thofe of the latter as 2 to 3. It was alfo manifeft, that the right-lines of the triangle were arithmetically proportional, or as the numbers 1, 2, 3, &c. therefore, the circles of the cone (being in a duplicate ratio of their diameters) as 1, 4, 9, &c. Alfo the circles of the parabolic conoid (being in the duplicate ratio of the ordinates, that is, in the ratio of the diameters) were as 1, 2, 3, &c. therefore their diameters as $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, &c. He therefore hoped, that from knowing the ratio of a feries of circles or fquares (which is the fame thing) to as many equals, he fhould be able to difcover what was the ratio of their diameters or fides to as many equals; and that if this were once proved univerfally, the quadrature of the circle would follow of confequence. For as it was already known, that all the parallel circles in a fphere were to as many in a cylinder as 2 to 3; if it could be from thence difcovered what was the ratio of the fum of all the diameters of the one to the fum of all the diameters of the other, the quadrature of the circle would be attained; as the former fum conftitutes the area of the circle, and the latter the area of the circumfcribed fquare: the geometrical problem being thus reduced to one purely arithmetical. Obferving then the analogy between the terms of certain infinite feries, and the ordinates of certain curves, he difcovered rules for finding the fums of thefe feries, and confequently attained the quadrature of thofe curves whofe ordinates were proportional to the terms of thefe feries. In

this

this enquiry he began with the more ſimple ſeries, firſt conſidering arithmetical progreſſions; then he proceeded to thoſe whoſe terms were as the ſquares, cubes, biquadrates, &c. or as the ſquare roots, cube roots, &c. of the terms of thoſe arithmetical progreſſions. He afterwards conſidered thoſe progreſſions whoſe terms were as any dimenſion whatſoever of the terms of the arithmetical progreſſions; that is, the indices or exponents of whoſe dimenſions were as any numbers, integral, fractional, or ſurd, whether poſitive or negative. He conſidered theſe progreſſions as conſiſting of an infinite number of terms, the laſt term, which repreſented the laſt ordinate of the curve, being ſtill finite; and the intermediate terms from o to the laſt, being infinite in number, repreſented ordinates applied to the axis, at infinitely ſmall and equal diſtances, between the vertex and laſt ordinate. Or perhaps theſe terms repreſented any other lines right or curve; or any plain or curve ſurfaces, in the caſe of ſolids, which were proportional to them. At length, by an induction of particulars, he came to this general theorem, which is the 64th of his Arithmetica Infinitorum, "In any infinite "ſeries of quantities beginning from o, and continually increaſ-"ing according to any power whatſoever, whether ſimple or "compounded of ſuch as are ſimple, the ratio of all the terms "of ſuch a ſeries is to as many times the greateſt, as unity to the "index of that power increaſed by unity." And this is the ſame in ſubſtance with Sir I. Newton's firſt rule for the quadrature of ſimple curves, in his Anal. per Equat. t. n. infin. which was inveſtigated, in the manner juſt now mentioned, by an induction of particulars by Wallis, but which Newton demonſtrated univerſally by an indefinite index, as was his manner, comprehending, in one general propoſition, all thoſe particular caſes which

others had demonftrated with limitations. In the analytical demonftration however which he has given of this thorem, certain quantities are omitted as being indefinitely little; and therefore it is not delivered with that ακριβεια which is required in fubjects of this nature. Fermat has given us a fynthetic demonftration of this theorem, which Doctor Horfely has inferted in his notes on this tract of Newton, but it is fo tedious and prolix, that even the analytical is preferable to it. I fhall here give a fynthetical demonftration alfo of the fame general propofition on the principles of *prime and ultimate ratios*, a method of reafoning which Newton feems to have had fome idea of even at the time of his writing this Analyfis, in the year 1669, though probably he did not bring it to perfection until eighteen years after, when he firft publifhed the Principia.

RULE I. QUAD. *of* SIMPLE CURVES.

Plate III. Fig. 2. LET the bafe AB of any curve AD have BD for its perpendicular ordinate; and let $AB=X$, $BD=Y$; alfo let a be a given quantity, and m, n, whole numbers. Then if $Y=aX^{\frac{m}{n}}$, it fhall be, area $ABD=\frac{n}{m+n}X^{\frac{m+n}{n}}$.

LET DC, AC, drawn through D and A parallel to AB and DB, meet in C; draw the ordinate d b indefinitely near to DB, meeting CD in s; and through d draw r p parallel to AB. Since $Y=aX^{\frac{m}{n}}$, the moment of Y will be equal to the moment of

of $aX^{\frac{m}{n}}$; that is, $y = \frac{am}{n} X^{\frac{m-n}{n}} x$, by Caf. 5. Lem. 2. B. 2. Principia. Now the rectangle sDBb is the rectangle under the ordinate and moment of the abfciffe, that is, Yx; and the rectangle CDpr is the rectangle under the abfciffe and moment of the ordinate, that is, Xy; therefore bD : Cp :: Yx : Xy; that is, fubftituting its value for y,

$$\text{as } Yx : \frac{am}{n} X^{\frac{m}{n}} x;$$

$$\text{or as } Y : \frac{am}{n} X^{\frac{m}{n}};$$

$$\text{or as } aX^{\frac{m}{n}} : \tfrac{am}{n} X^{\frac{m}{n}}, \text{ that is, as } 1 \text{ to } \frac{m}{n}.$$

By a like procefs it may be fhewn, that if AB be divided into an indefinite number of parts, and upon each there be conftructed a rectangle ob in the fame manner as on bB, and through g there be drawn tq parallel to the bafe; the rectangle ob will be to the correfponding rectangle rq ultimately in the given ratio of 1 to $\frac{m}{n}$. Therefore the fum of all the indefinitely little rectangles sB will be to the fum of the correfponding rectangles Cp, in the fame ratio; therefore the curve ADB is to the curve ACD as 1 to $\frac{m}{n}$, Cor. Lem. 4. B. 1. Prin. and the curve ADB to the rectangle ACDB, as 1 to $\frac{m+n}{n}$; but the rectangle $ACDB = YX = aX^{\frac{m+n}{n}}$; there-

fore the curve ADB is to $aX^{\frac{m+n}{n}}$ as 1 to $\frac{m+n}{n}$, and therefore equal to $\frac{n}{m+n} \times aX^{\frac{m+n}{n}}$. Q. E. D.

If the ordinate BD be oblique to the baſe, the area, found as above, muſt be diminiſhed in the ratio of radius to the ſine of the angle made by the ordinate and baſe.

This demonſtration being admitted, the whole doctrine of quadratures becomes a branch of *prime and ultimate ratios*, and conſequently of pure geometry.

We are to obſerve, that the reaſon why the curves treated of above are perfectly quadrable is, becauſe the rectangles inſcribed in the curve are to the reſpective rectangles inſcribed in the exterior ſpace, ultimately, in a given ratio, whence the curve will be to that ſpace (Cor. Lem. 4. Prin.) and conſequently to the circumſcribed rectangle, in a given ratio. But this is not the caſe in the circle, which therefore is not quadrable by this method, at leaſt in its preſent ſtate. But though the ratio of the rectangles inſcribed in a quadrant to their correſponding rectangles in the exterior ſpace of a circumſcribed ſquare perpetually varies from the beginning of the quadrant to the end, yet this variation is regular, beginning from the finite ratio of 2 to 1, and conſtantly approaching the infinite ratio of 1 to nothing. The law of which approach may be thus determined:

If

IF a ſquare APCO be circumſcribed about a quadrant ACO, and the radius AO be divided into any number of equal parts whatſoever, and on theſe parts rectangles as moBD de erected, and inſcribed in the quadrant, and through the extremities of the ordinates nm, BD, right lines tp. rB be drawn parallel to the radius AO, and thus as many correſponding rectangles, as rBtp, be inſcribed in the exterior ſpace: then the breadth of theſe rectangles being diminiſhed indefinitely, each rectangle in the quadrant will be to its correſponding rectangle in the exterior ſpace as DR to DO, that is, as the ſum of the radius AO and the ſegment DO between the centre and ordinate to that ſegment. Plate III. Fig. 3.

LET the abſciſſe $AD=X$; the ordinate $BD=Y$; the diameter $AR=a$.

$Y^2=\overline{aX-X^2}$, from the nature of the circle.

Therefore $2Yy=ax-2Yx$ (Lem. 2. B. 2. Prin.), $y=\dfrac{ax-2Xx}{2\overline{aX-X^2}|^{\frac{1}{2}}}$;

but moBD is to Brpt as Yx to Xy.

$$\text{that is, as } Yx \text{ to } \frac{aXx-2X^2x}{2\overline{aX-X^2}|^{\frac{1}{2}}}$$

$$\text{that is, as } Y \text{ to } \frac{aX-2X^2}{2\overline{aX-X^2}|^{\frac{1}{2}}}$$

$$\text{that is, as } \overline{aX-X^2}|^{\frac{1}{2}} \text{ to } \frac{aX-2X^2}{2\overline{aX-X^2}|^{\frac{1}{2}}}$$

$$\text{that is, as } 2aX-2X^2 \text{ to } aX-2X^2$$

$$\text{or, as } a-X \text{ to } \tfrac{1}{2}a-X, \text{ or as DR to DO.}$$

IN

In A, the extremity of the diameter, DR is to DO as 2 to 1; as D approaches O, this ratio continually encreafes, and in O this ratio becomes 1 to nothing.

Hence if the radius AO be divided into any number of equal parts, and there be conftituted a feries of fractions, whofe numerators are the natural numbers increafing from unity to that number of parts, and whofe denominators are the continuation of that feries; then the rectangles infcribed in the circle will be to the refpective rectangles in the exterior fpace, ultimately as 1 to the fucceffive terms of this feries beginning with the leaft. Thus, fuppofe the radius divided into 8 equal parts, then the ultimate ratio of the correfponding rectangles from the beginning of the quadrant will be the ratio of 1 to the terms of the following feries:

$$\frac{8}{16},\ \frac{7}{15},\ \frac{6}{14},\ \frac{5}{13},\ \frac{4}{12},\ \frac{3}{11},\ \frac{2}{10},\ \frac{1}{9}.$$

On fimilar principles we may demonftrate the following theorem in the Analyfis Equationum, &c. in a much more fimple and elegant manner than by the method of fluxions.

Plate III. Fig. 4. Let ALE be an ellipfe, whofe $\frac{1}{2}$ tranfverfe axis is CL, $\frac{1}{2}$ conjugate AC; let CB=x, BD=y, AC=c, and CL=t. The ultimate ratio of DG : GH, is the ratio of DB : BT, $\therefore$ ultimately, $DG^2 : GH^2 :: Y^2 : BT^2$.

$y^2 = \frac{t^2}{c^2} \times \overline{c^2 - x^2}$ from the nature of the figure;

$BT = \frac{c^2}{x} - x$, and $BT^2 = \frac{\overline{c^2 - x^2}|^2}{x^2}$

$\because DG^2$

$$\because\ DG^2 = GH^2 \times \frac{t^2\,x^2}{c^4 - c^2\,x^2}$$

$$DH^2 = HG^2 + DG^2 = GH^2 + GH^2 \times \frac{t^2\,x^2}{c^4 - c^2\,x^2} = GH^2 \times \frac{c^4 - \begin{smallmatrix}c^2\\+t^2\end{smallmatrix}\,x^2}{c^4 - c^2\,x^2}$$

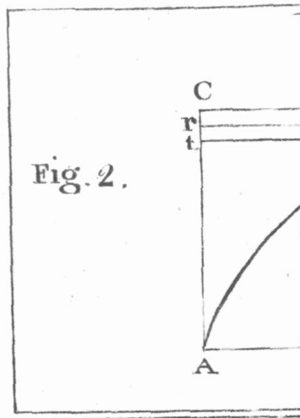
$\because\ DH = GH \times \dfrac{\sqrt{c^4 - \begin{smallmatrix}c^2\\+t^2\end{smallmatrix}\,x^2}}{\sqrt{c^4 - c^2\,x^2}}$; that is, ſuppoſing $c = 1$, $t^2 - c^2 = a$, and $c^2 = b$.

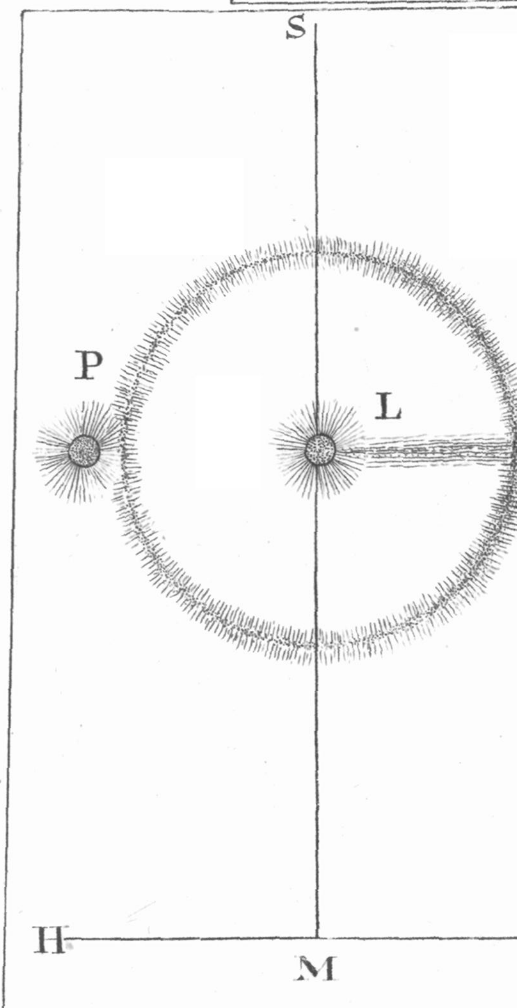
$DH = GH \times \dfrac{\sqrt{1 + ax^2}}{\sqrt{1 - bx^2}}$: or, the moment of the arch is equal to the moment of the abſciſſe multiplied into $\dfrac{\sqrt{1 + ax^2}}{\sqrt{1 - bx^2}}$.

Wherefore a curve, whoſe ordinate is this latter quantity, increaſes in the ſame manner as the elliptical arch ; and conſequently the area deſcribed by that ordinate is analogous to the length of the elliptical arch ; ſo that both may be denoted by the ſame algebraical expreſſion.

PLATE III

Fig. 2.





Let the drawing be

SYNTHETIC DEMONSTRATION &c.

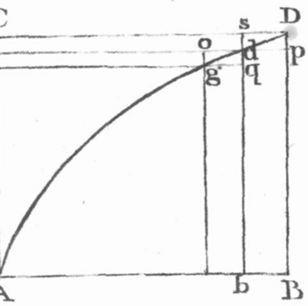


Fig. 3.

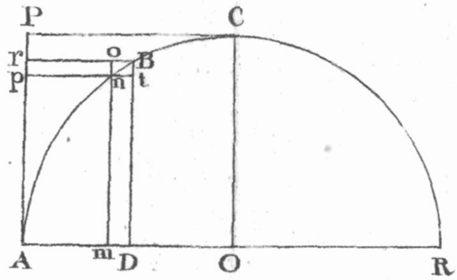
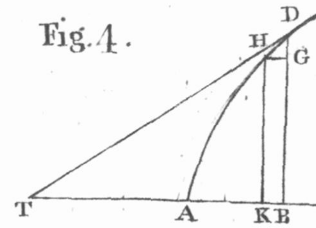


Fig. 4.



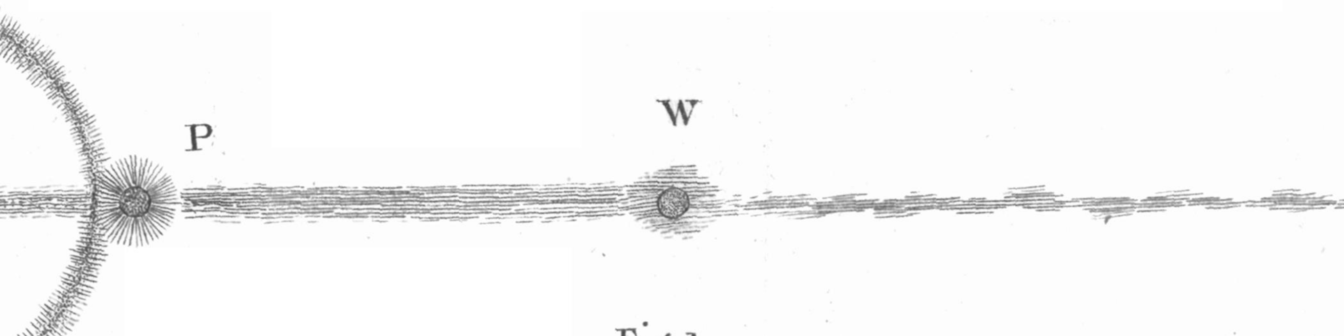
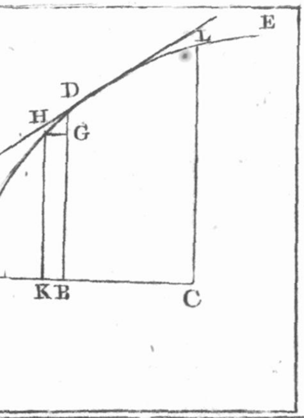
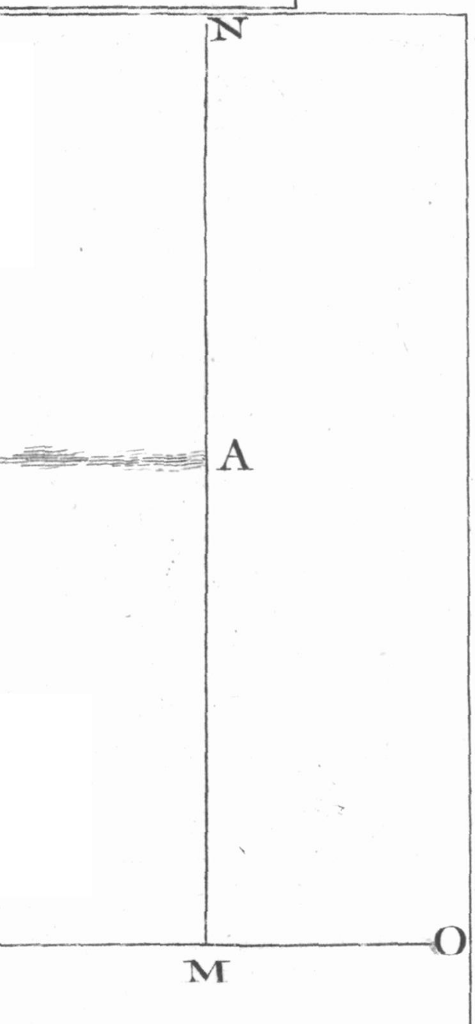


Fig. 1.

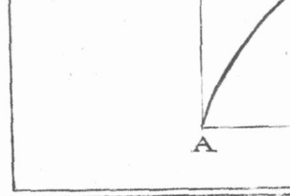
g be bent inwards till MM becomes a semicircle, then will SM represen

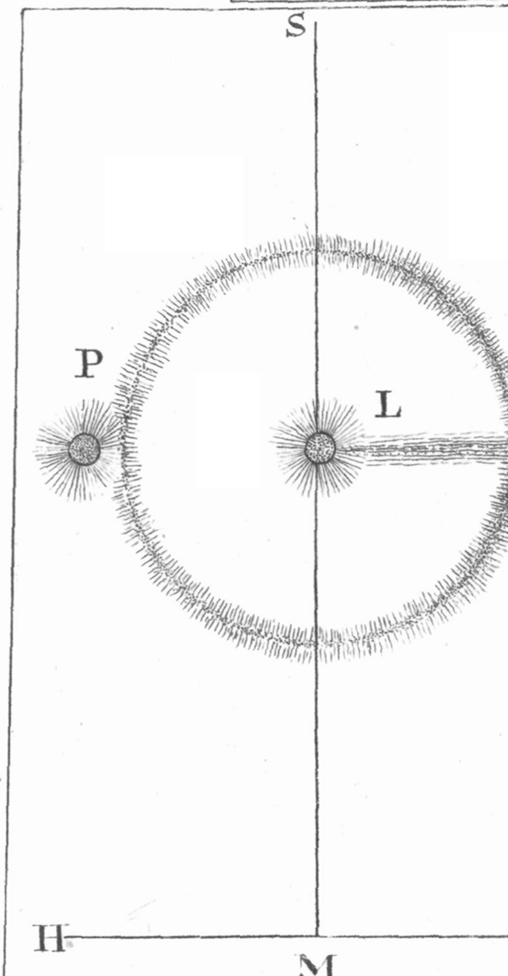
Sci.
To Face Page 40





...resent the South Meridian

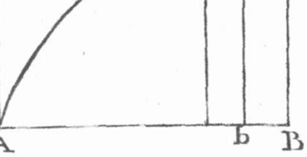


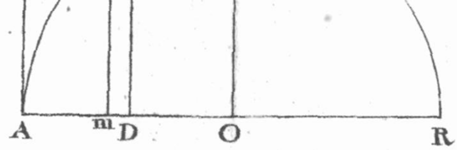


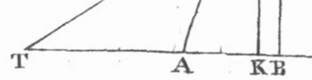
Let the drawing be b

NM *the North Meridian,*

true Sun, W *white part*







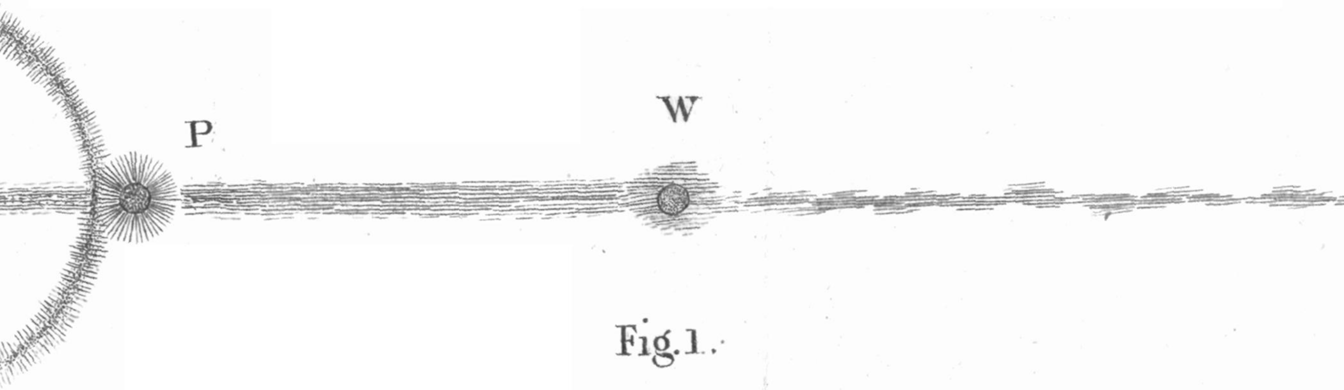
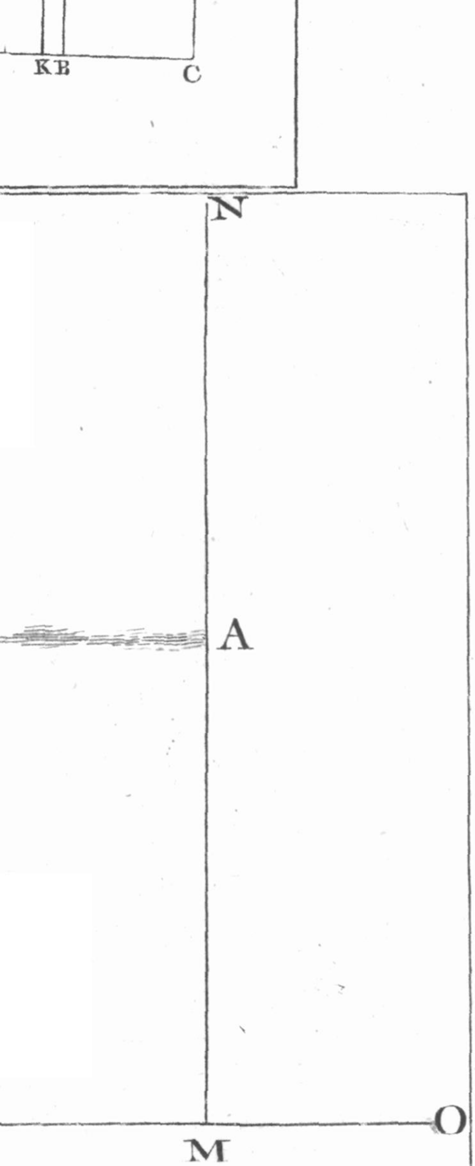


Fig. 1.

g be bent inwards till MM *becomes a semicircle, then will* SM *represen*

idian, PP *prismatic Circle with two prismatic suns or parhelia at 26° dis*

e parhelion at 90° dist. from the true Sun LA *luminous Almicantar and*



resent the South Meridian
26° dist. on each side the
and HO the Horizon.

I. Ford sculp.